Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M **12/05**

Numerical procedures in multiobjective optimization with variable ordering structures

Gabriele Eichfelder

Mai 2012

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69-3621
Fax: +49 3677 69-3270
http://www.tu-ilmenau.de/math/

Numerical procedures in multiobjective optimization with variable ordering structures

Gabriele Eichfelder[1]

May 21, 2012

Abstract

Multiobjective optimization problems with a variable ordering structure instead of a partial ordering have recently gained interest due to several applications. In the last years a basic theory has been developed for such problems. The difficulty in their study arises from the fact that the binary relations of the variable ordering structure, which are defined by a cone-valued map which associates to each element of the image space a pointed convex cone of dominated or preferred directions, are in general not transitive.

In this paper we propose numerical approaches for solving such optimization problems. For continuous problems a method is presented using scalarization functionals which allows the determination of an approximation of the infinite optimal solution set. For discrete problems the Jahn-Graef-Younes method known from multiobjective optimization with a partial ordering is adapted to allow the determination of all optimal elements with a reduced effort compared to a pairwise comparison.

Key Words: Multiobjective optimization, variable ordering structure, variable domination structure.

Mathematics subject classifications (MSC 2000): 90C29, 90C30.

[1]Institute of Mathematics, Ilmenau University of Technology, Weimarer Str. 25, D-98693 Ilmenau, Germany, email: `Gabriele.Eichfelder@tu-ilmenau.de`

1 Introduction

In the last few years multiobjective optimization problems with a variable ordering structure have gained interest motivated by several applications [1, 2, 3, 4, 5, 6]. Before, usually multiobjective optimization problems were considered with the image space partially ordered by a convex cone. Replacing this partial ordering by other binary relations allows the treatment of multiobjective optimization problems where the cone of preferred or dominated directions in the image space depends on the actual element in the image space. This is mathematically modeled by a set-valued map which associates to each element of the space a pointed convex cone. Based on this map, which is called ordering map, two binary relations can be defined depending on the choice of the cone w.r.t. which the elements are compared.

We show that both binary relations are in general not transitive without additional assumptions. The loss of transitivity is the main drawback when trying to generalize known numerical methods developed in partially ordered spaces to multiobjective optimization problems with a variable ordering structure. We present a necessary condition for optimality as well as scalarization results which can be combined to a numerical algorithm for continuous multiobjective optimization problems with a variable ordering structure. Additionally, we adapt an algorithm for discrete problems in partially ordered spaces, the Jahn-Graef-Younes method, to variable ordering structures.

In the literature hardly any numerical procedures solving multiobjective optimization problems with variable ordering structures exist. Wacker presented in [4] an algorithm for solving such continuous problems in medical image registration where the algorithm is especially designed for this application only. Furthermore, he determines only one optimal solution while we aim on determining or at least approximating the whole image set of optimal solutions. For the notion of equitability a numerical procedure based on evolutionary algorithms was presented by Shukla et al. [7]. Equitability is a stronger concept than efficiency w.r.t. the natural ordering cone, i.e. the equitable efficient elements are a subset of the set of efficient elements w.r.t. the componentwise (natural) partial ordering. This notion corresponds to the optimality notions w.r.t. a variable ordering structure with the images of the cone-valued ordering map being constant for all elements within a so-called sector and the space being partitioned in a finite number of sectors [1].

Hirsch et al. present in [8] a numerical method for determining an approximation of the set of optimal elements w.r.t. a variable ordering structure for compact sets and using the assumption that the natural ordering cone is included in each image of the ordering map. Using known evolutionary algorithm for multiobjective optimization problems an approximation of the set of efficient elements w.r.t. the natural ordering cone is determined and among this finite approximation set the optimal elements of this approximation set w.r.t. the variable ordering are selected. This selection is considered to be an approximation of the set of optimal elements w.r.t. the variable ordering structure. However, this selection may contain elements which are not even weakly optimal solutions w.r.t. the variable ordering structure.

In Section 2 we present the two binary relations and accordingly the two optimality notions – the nondominated and the minimal elements – w.r.t. a variable ordering structure. Further, we study the properties of the two binary relations like transitivity. In Section 3 the theoretical background for the algorithm is developed: characterizations of the optimal elements, necessary conditions and scalarization results are proven. Section 4 and 5 are devoted to the numerical procedures for continuous and discrete sets, respectively. Finally, in Section 6, the algorithm are applied on some test instances derived from known test instances for multiobjective optimization problems in partially ordered spaces. We end with some concluding remarks.

2 Variable Ordering Structures

In multiobjective optimization one considers optimization problems

$$\min_{x \in S} f(x) \tag{1}$$

with an objective function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and $S \subset \mathbb{R}^n$ a nonempty set. A point $\bar{x}$ is denoted an optimal solution of (1) if $\bar{y} := f(\bar{x})$ is an optimal element of the set $f(S)$.

For the definition of an optimal (or efficient) element of the image set $f(S)$ it is usually assumed that a partial ordering $\geq_K$ in $\mathbb{R}^m$ is given by a nontrivial pointed convex cone $K \subset \mathbb{R}^m$, which is then also called an ordering cone. Recall that a set K is called a cone if $\lambda x \in K$ for all $\lambda \geq 0$ and $x \in K$. And a cone is convex if $K + K \subset K$. A cone satisfying $K \cap (-K) = \{0\}$ is called pointed. We speak of the natural (componentwise) ordering if $K = \mathbb{R}^m_+$. We write $x \leq_K y$ for $y - x \in K$ and denote an element $\bar{y} \in A := f(S)$ an efficient element of A w.r.t. K if

$$(\{\bar{y}\} - K) \cap A = \{\bar{y}\}. \tag{2}$$

The set of efficient elements is denoted by $\mathcal{E}_K$. Condition (2) is equivalent to that there is no other point $y \in A$ with

$$\bar{y} \in \{y\} + K. \tag{3}$$

Replacing the partial ordering by a variable ordering structure, it is assumed that a set-valued map $\mathcal{D}\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ with $\mathcal{D}(y)$ a nontrivial pointed convex cone for all $y \in \mathbb{R}^m$ is given. This map defines an ordering cone for each point in the image space. Using the cone-valued map $\mathcal{D}$, two different binary relations are defined by

$$y \leq_1 \bar{y} \;\text{ if }\; \bar{y} \in \{y\} + \mathcal{D}(y) \tag{4}$$

and by

$$y \leq_2 \bar{y} \;\text{ if }\; \bar{y} \in \{y\} + \mathcal{D}(\bar{y}). \tag{5}$$

The first relation mean that a point $\bar{y}$ is worse than y if $\bar{y}$ is dominated by y. According to the second relation, an element y is better than another element $\bar{y}$ if $y \in \{\bar{y}\} - \mathcal{D}(\bar{y})$, which means that y is preferred to $\bar{y}$. One speaks of a variable ordering (structure) given by the ordering map $\mathcal{D}$, even though the binary relations given above are in general neither transitive nor compatible with the linear structure of the space as we show in the following. We express thereby that the partial ordering given by a convex cone is replaced by $\mathcal{D}$.

The two relations lead to the following two optimality notions. The first, based on (4), origins from Yu [9] and corresponds to the reformulation of the classical concept in (3):

Definition 2.1. *An element $\bar{y} \in A$ is a* nondominated element *of the set A w.r.t. the ordering map $\mathcal{D}$, if no other $y \in A$ exists such that*

$$\bar{y} \in \{y\} + \mathcal{D}(y).$$

The following notion based on (5) origins from Chen et al. [10]:

Definition 2.2. *A point $\bar{y} \in A$ is called a* minimal element *of the set A w.r.t. the ordering map $\mathcal{D}$ if*

$$(\{\bar{y}\} - \mathcal{D}(\bar{y})) \cap A = \{\bar{y}\}.$$

Thus, if there is no y in A, which is preferred to $\bar{y}$, i.e. such that $y \in \{\bar{y}\} - (\mathcal{D}(\bar{y}) \setminus \{0\})$, then $\bar{y}$ is called a minimal element of A w.r.t. $\mathcal{D}$. Following the concepts mentioned in the beginning of this section, a point $\bar{x}$ is called a minimal/nondominated solution of the multiobjective optimization problem (1) if $\bar{y} := f(\bar{x})$ is a minimal/nondominated element of the set $A := f(S)$.

Replacing in (2) the cone K by $\mathrm{int}(K) \cup \{0\}$ with $\mathrm{int}(K)$ denoting the interior of K and thereby assuming the interior to be nonempty, we obtain a weaker optimality notion which is called weakly efficient. Analogously, assuming $\mathrm{int}(\mathcal{D}(y)) \neq \emptyset$ whenever considered, an element $\bar{y} \in A$ is called a weakly nondominated element of A if there is no $y \in A$ with

$$\bar{y} \in \{y\} + \mathrm{int}(\mathcal{D}(y)),$$

and it is called a weakly minimal element of A if

$$(\{\bar{y}\} - \mathrm{int}(\mathcal{D}(\bar{y}))) \cap A = \emptyset.$$

Next we examine the properties of the two binary relations $\leq_1$ and $\leq_2$.

Lemma 2.1. ***(a)*** *The relations defined in (4) and (5) are reflexive.*

(b) *The binary relation $\leq_1$ defined in (4) is transitive if the condition*

$$\mathcal{D}(y + d) \subset \mathcal{D}(y) \text{ for all } y \in \mathbb{R}^m \text{ and for all } d \in \mathcal{D}(y) \tag{6}$$

is satisfied. If $\mathcal{D}(y)$ is closed for all $y \in \mathbb{R}^m$, then (6) also is necessary for the transitivity of $\leq_1$.

(c) *The binary relation $\leq_2$ defined in (5) is transitive if the condition*

$$\mathcal{D}(y - d) \subset \mathcal{D}(y) \text{ for all } y \in \mathbb{R}^m \text{ and for all } d \in \mathcal{D}(y) \tag{7}$$

is satisfied. If $\mathcal{D}(y)$ is closed for all $y \in \mathbb{R}^m$, then (7) also is necessary for the transitivity of $\leq_2$.

(d) *Any of the two relations $\leq_1$ or $\leq_2$ is compatible with addition if and only if $\mathcal{D}$ is a constant map.*

(e) *Any of the two relations $\leq_1$ or $\leq_2$ is compatible with nonnegative scalar multiplication if and only if*

$$\mathcal{D}(y) \subset \mathcal{D}(\alpha\, y) \quad \textit{for all} \quad y \in \mathbb{R}^m \quad \textit{and for all} \quad \alpha > 0. \tag{8}$$

(f) *The relations defined in (4) and (5) are antisymmetric if the cone $\mathcal{D}(\mathbb{R}^m) = \bigcup_{y\in\mathbb{R}^m} \mathcal{D}(y)$ is pointed.*

Proof. (a) The relations are both reflexive as the sets $\mathcal{D}(y)$ are assumed to be cones and thus $0 \in \mathcal{D}(y)$ for all $y \in \mathbb{R}^m$.

(b) We first show that the condition (6) is sufficient. As $x \leq_1 y$ and $y \leq_1 z$ correspond to $y - x \in \mathcal{D}(x)$ and $z - y \in \mathcal{D}(y)$, (6) implies $\mathcal{D}(y) \subset \mathcal{D}(x)$ and we get $z - x = (z - y) + (y - x) \in \mathcal{D}(y) + \mathcal{D}(x) \subset \mathcal{D}(x)$ and hence $x \leq_1 z$ for arbitrary $x, y, z \in \mathbb{R}^m$.

Next we show that condition (6) also is necessary if $\mathcal{D}(y)$ is closed for all $y \in \mathbb{R}^m$. For that we assume $\leq_1$ to be transitive, but (6) does not hold. Then there exists some $x \in \mathbb{R}^m$ and some $d \in \mathcal{D}(x)$ as well as some

$$k \in \mathcal{D}(x + d) \setminus \{0\} \quad \text{with} \quad k \notin \mathcal{D}(x). \tag{9}$$

For all $s > 0$ we obtain $sk \in \mathcal{D}(x + d) \setminus \{0\}$ and $sk \notin \mathcal{D}(x)$. We set

$$y := x + d \quad \text{and} \quad z_s := y + sk = x + d + sk \quad \text{for all} \quad s > 0.$$

Then $y - x = d \in \mathcal{D}(x)$ and $z_s - y = sk \in \mathcal{D}(x + d) = \mathcal{D}(y)$ for all $s > 0$. Because $\leq_1$ is transitive, it holds $z_s - x = d + sk \in \mathcal{D}(x)$ for all $s > 0$, i.e. $\frac{1}{s}d + k \in \mathcal{D}(x)$ for $s > 0$ implying, because $\mathcal{D}(x)$ is closed, $k \in \mathcal{D}(x)$ in contradiction to (9).

(c) We first show that the condition (7) is sufficient. As $x \leq_2 y$ and $y \leq_2 z$ correspond to $y - x \in \mathcal{D}(y)$ and $z - y \in \mathcal{D}(z)$, (7) implies $\mathcal{D}(y) \subset \mathcal{D}(z)$ and we get $z - x = (z - y) + (y - x) \in \mathcal{D}(z) + \mathcal{D}(y) \subset \mathcal{D}(z)$ and hence $x \leq_1 z$ for arbitrary $x, y, z \in \mathbb{R}^m$.

Next we show that condition (7) also is necessary if $\mathcal{D}(y)$ is closed for all $y \in \mathbb{R}^m$. For that we assume $\leq_2$ is transitive, but (7) does not hold. Then there exists some $z \in \mathbb{R}^m$ and some $d \in \mathcal{D}(z)$ as well as some $k \in \mathbb{R}^m \setminus \{0\}$ with

$$sk \in \mathcal{D}(z - d) \setminus \{0\} \quad \text{and} \quad sk \notin \mathcal{D}(z) \quad \text{for all } s > 0. \tag{10}$$

We set $y := z - d$ and $x_s := y - sk = z - d - sk$ for $s > 0$. Then $y - x_s = sk \in \mathcal{D}(z - d) = \mathcal{D}(y)$ and $z - y = d \in \mathcal{D}(z)$ for all $s > 0$. Because $\leq_2$ is transitive, it holds $z - x_s = d + sk \in \mathcal{D}(z)$ for all $s > 0$ implying $k \in \mathcal{D}(z)$ in contradiction to (10).

(d) Compatibility with addition corresponds for both relations to the property $\mathcal{D}(y) + \mathcal{D}(z) \subset \mathcal{D}(y + z)$ for any $y, z \in \mathbb{R}^m$, i.e. to the subadditivity of the cone-valued map $\mathcal{D}$. Using for instance [11, Lemma 2.21] the conclusion follows.

(e) As $\mathcal{D}(y)$ is a cone for all $y \in \mathbb{R}^m$ it holds $\mathcal{D}(y) = \alpha\mathcal{D}(y)$ for all $\alpha > 0$ and thus compatibility with nonnegative scalar multiplication corresponds for both relations to the property $\mathcal{D}(y) \subset \mathcal{D}(\alpha\, y)$ for all $y \in \mathbb{R}^m$ and all $\alpha > 0$.

(f) $y \leq_1 z$ and $z \leq_1 y$ are equivalent to $z \in \{y\} + \mathcal{D}(y)$ and $z \in \{y\} - \mathcal{D}(z)$, thus $z - y \in \mathcal{D}(\mathbb{R}^m) \cap (-\mathcal{D}(\mathbb{R}^m))$, i.e. $z = y$. Analogously for $\leq_2$. $\square$

3 Characterization of Optimal Elements

In this section necessary and sufficient conditions for (weakly) nondominated and (weakly) minimal elements are given. Some of these characterizations are based on a new nonlinear scalarization functional which generalizes a well-studied functional known as nonconvex separational functional by Gerstewitz (Tammer) [12] or smallest monotone map [13]: $\psi_{a,r}\colon \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$,

$$\psi_{a,r}(y) := \inf\{t \in \mathbb{R} \mid a + t\,r - y \in K\} \quad \text{for all} \quad y \in \mathbb{R}^m \tag{11}$$

with $K \subset Y$ a convex cone, $a, r \in \mathbb{R}^m$. This functional was used in vector optimization by Pascoletti and Serafini [14] and was already studied by Rubinov [15] and Krasnoleski. Its properties are well studied, see for instance [16, Theorem 2.3.1, Corollary 2.3.5], [17, Prop. 2.1] and [18]. If $K \subset \mathbb{R}^m$ is a nontrivial closed pointed convex cone and $r \in K$ then $\psi_{a,r}$ is lower semicontinuous and convex. If, additionally, $r \in \operatorname{int}(K)$, then $\psi_{0,r}$ is sublinear, the function $\psi_{a,r}$ is continuous and finite valued and $\psi_{a,r}(y) = \min\{t \in \mathbb{R} \mid a + t\,r - y \in K\}$ for all $y \in \mathbb{R}^m$.

We start with necessary optimality conditions relating nondominated/minimal elements with efficient elements. As before, let A be a nonempty subset of $\mathbb{R}^m$ and $\mathcal{D}\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ a set-valued map with $\mathcal{D}(y)$ a nontrivial pointed convex cone for all $y \in \mathbb{R}^m$. If not mentioned otherwise, $K \subset \mathbb{R}^m$ is assumed to be a pointed convex cone.

Lemma 3.1. ***(a)*** *Any nondominated element of A w.r.t. $\mathcal{D}$ is also an efficient element of A with $\mathbb{R}^m$ partially ordered by a convex cone K with $K \subset \bigcap_{y\in A} \mathcal{D}(y)$.*

(b) *An element $\bar{y}$ of A is a (weakly) minimal element of A w.r.t. $\mathcal{D}$ if and only if it is a (weakly) efficient element of A with $\mathbb{R}^m$ partially ordered by $K := \mathcal{D}(\bar{y})$.*

(c) *Any minimal element of A w.r.t. $\mathcal{D}$ is also an efficient element of A with $\mathbb{R}^m$ partially ordered by a convex cone K with $K \subset \bigcap_{y\in A} \mathcal{D}(y)$.*

Proof. (a) According to the definition, $\bar{y}$ nondominated of A w.r.t. $\mathcal{D}$ is equivalent to $\bar{y} \notin \{y\} + \mathcal{D}(y) \setminus \{0\}$ for all $y \in A$, and hence $\bar{y} \notin \{y\} + K \setminus \{0\}$ for all $y \in A$ and any K with $K \subset \mathcal{D}(y)$ for all $y \in A$.

(b) follows from the definition.

(c) According to (b) any minimal element of A w.r.t. $\mathcal{D}$ is also an efficient element of A with $\mathbb{R}^m$ partially ordered by $\mathcal{D}(\bar{y})$ and thus also if partially ordered by $K \subset \mathcal{D}(\bar{y})$. □

This simple lemma delivers a useful necessary condition for determining a subset of the set A which contains all minimal and all nondominated elements w.r.t. a variable ordering structure. Note that for instance in [3] ordering maps were proposed for modeling variable preferences of decision makers with $\mathbb{R}^m_+ \subset \mathcal{D}(y)$ for all $y \in A$, see the following example. Thus in that case we can choose $K := \mathbb{R}^m_+ \subset \bigcap_{y\in A} \mathcal{D}(y)$.

Example 3.1. *Consider the cone-valued map $\mathcal{D}\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ defined on some set $A \subset \mathbb{R}^m$, which is assumed to be bounded from below, and given by*

$$\mathcal{D}(y) := \left\{d \in \mathbb{R}^m \mid d^\top(y-p) \geq \gamma \cdot \|d\|_2 \cdot [y-p]_{\min}\right\} \quad \text{for all } \ y \in A \tag{12}$$

where $\gamma \in (0,1]$, $p_i < \inf_{y\in A} y_i$ for $i = 1,\ldots,m$, and $[y-p]_{\min} := \min_{i=1,\ldots,m} y_i - p_i$, compare [3, 8]. Then $\mathbb{R}^m_+ \subset \mathcal{D}(y)$ for all $y \in A$, because for any $d \in \mathbb{R}^m_+ \setminus \{0\}$ it holds

$$\frac{d^\top(y-p)}{\gamma\|d\|_2[y-p]_{\min}} \geq \frac{d^\top(y-p)}{\gamma\|d\|_1[y-p]_{\min}} \geq \frac{\|d\|_1[y-p]_{\min}}{\gamma\|d\|_1[y-p]_{\min}} = \frac{1}{\gamma} \geq 1,$$

i.e. $d \in \mathcal{D}(y)$. Note that the cones $\mathcal{D}(y)$ are Bishop-Phelps cones [19]: by defining

$$\ell(y) := \frac{y-p}{[y-p]_{\min}\cdot\gamma}$$

we can write

$$\mathcal{D}(y) = \{d \in \mathbb{R}^m \mid \ell(y)^\top d \geq \|d\|_2\}. \tag{13}$$

For the results in Lemma 3.1 we need that the intersection of all cones $\mathcal{D}(y)$ with $y \in A$ is nontrivial. Later, we even assume that this intersection has a nonempty interior. However, without this assumption it might even occur that weakly nondominated elements lie within the interior of the set A.

Example 3.2. *Let $A = [1,3] \times [1,3]$, and $\mathcal{D}\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$,*

$$\mathcal{D}(y_1,y_2) := \begin{cases} \mathbb{R}^2_+ & \text{if } y_1 \geq 2\ , \\ \{(z_1,z_2) \in \mathbb{R}^2 \mid z_1 \leq 0,\ \ z_2 \geq 0\} & \text{otherwise.} \end{cases}$$

Then $\bar{y} = (2,2)$ is a weakly nondominated element of A w.r.t. $\mathcal{D}$ but $\bar{y} \notin \partial A$.

See [20, Example 2.1] for an example with $\operatorname{int}\left(\bigcap_{y\in A}\mathcal{D}(y)\right) = \emptyset$ where no optimal element w.r.t. the variable ordering structure exists at all.

Lemma 3.2. *Let int($\mathcal{D}(y)$) be nonempty for all $y \in A$.*

(a) *If $\bigcap_{y\in A}$ int($\mathcal{D}(y)$) $\neq \emptyset$ and $\bar{y} \in A$ is a weakly nondominated element of A w.r.t. $\mathcal{D}$, then $\bar{y} \in \partial A$.*

(b) *If $\bar{y} \in A$ is a weakly minimal element of A w.r.t. $\mathcal{D}$, then $\bar{y} \in \partial A$.*

Proof. (a) We assume $\bar{y} \in \operatorname{int}(A)$. Let $d \in \bigcap_{y\in A}\operatorname{int}(\mathcal{D}(y))$. Then $d \neq 0$ and there exists $\lambda > 0$ with $\bar{y} - \lambda d \in A \setminus \{\bar{y}\}$. As

$$-\lambda d \in -\bigcap_{y\in A}\operatorname{int}(\mathcal{D}(y)) \subset -\operatorname{int}(\mathcal{D}(\bar{y}-\lambda d))$$

we have $\bar{y} - \lambda d \in A \cap (\{\bar{y}\} - \operatorname{int}(\mathcal{D}(\bar{y}-\lambda d)))$ or $\bar{y} \in \{\bar{y}-\lambda d\} + \operatorname{int}(\mathcal{D}(\bar{y}-\lambda d))$, being a contradiction to $\bar{y}$ weakly nondominated.

(b) Follows directly from Lemma 3.1(b) and the known fact that in partially ordered spaces all weakly efficient elements are a subset of the boundary of the set [21, Theorem 1.13]. However, it can also be shown easily by choosing any $d \in \operatorname{int}(\mathcal{D}(\bar{y}))$. Then the proof is analogous to the proof of part (a). $\square$

The necessary condition of Lemma 3.1 is under some additional assumptions also sufficient. We need the notion of external stability which is also denoted domination property, see [13] and the references therein.

Definition 3.1. *Let the space $\mathbb{R}^m$ be partially ordered by some convex cone K and let $\mathcal{E}$ be a nonempty subset of the set $A \subset \mathbb{R}^m$. Then $\mathcal{E}$ is said to be* externally stable *if for all $y \in A \setminus \mathcal{E}$ there exists some $\bar{y} \in \mathcal{E}$ such that $y \in \{\bar{y}\} + K$.*

Thus, external stability holds if for all $y \in A$ there exists some $\bar{y} \in \mathcal{E}$ such that $y \in \{\bar{y}\} + K$. In [22, Section 3.2] and [13] conditions ensuring the external stability of the set of efficient elements are given. According to Theorem 3.2.10 in [22], if K is a closed pointed convex cone and A is K-compact, i.e. the sets $(\{y\} - K) \cap A$ are compact for all $y \in A$, then the set of efficient elements $\mathcal{E}_K$ of A w.r.t. the partial ordering introduced by K is externally stable, i.e. $A \subset \mathcal{E}_K + K$. For instance, if A is compact, then it is also K-compact for any closed cone K. Now we give the announced theorem.

Theorem 3.1. *Let $K \subset \bigcap_{y \in A} \mathcal{D}(y)$ be a pointed convex cone and denote by $\mathcal{E}_K$ the set of efficient elements of A w.r.t. K. Let $\mathcal{E}_K$ be externally stable and let*

$$y^1 \in \{y^2\} + K \textit{ imply } \mathcal{D}(y^1) \subset D(y^2) \textit{ for all } y^1, y^2 \in A. \tag{14}$$

Then $\bar{y} \in A$ is a nondominated element of A w.r.t. $\mathcal{D}$ if and only if $\bar{y}$ is a nondominated element of $\mathcal{E}_K$ w.r.t. $\mathcal{D}$.

Proof. By Lemma 3.1(a) and as $\mathcal{E}_K \subset A$, the condition is necessary. To show that it is also sufficient, assume $\bar{y}$ is a nondominated element of $\mathcal{E}_K$ but not of A w.r.t. $\mathcal{D}$. Then there exists some $y \in A \setminus \mathcal{E}_K$ with $\bar{y} \in \{y\} + \mathcal{D}(y) \setminus \{0\}$. As $y \in A \setminus \mathcal{E}_K$ and $\mathcal{E}_K$ is externally stable, there exists some $\hat{y} \in \mathcal{E}_K$ with $y \in \{\hat{y}\} + K \setminus \{0\}$. Condition (14) implies $\mathcal{D}(y) \subset \mathcal{D}(\hat{y})$ and we obtain

$$\bar{y} \in \{\hat{y}\} + K + \mathcal{D}(y) \subset \{\hat{y}\} + \mathcal{D}(y) \subset \{\hat{y}\} + \mathcal{D}(\hat{y})$$

and $\bar{y} \neq \hat{y} \in \mathcal{E}_K$ in contradiction to $\bar{y}$ a nondominated element of $\mathcal{E}_K$ w.r.t. $\mathcal{D}$. □

The following theorem shows that condition (14) is satisfied if the binary relation $\leq_1$ defined by the ordering map is transitive.

Lemma 3.3. *Let $\mathcal{D}(y)$ be closed for all $y \in \mathbb{R}^m$ and let $K \subset \bigcap_{y \in A} \mathcal{D}(y)$. If the binary relation $\leq_1$ is transitive, then (14) is satisfied.*

Proof. The relation $\leq_1$ is transitive according to Lemma 2.1(b) if and only if $\mathcal{D}(y + d) \subset \mathcal{D}(y)$ for all $y \in Y$ and for all $d \in \mathcal{D}(y)$. For $y^1 \in \{y^2\} + K$ it holds $y^1 = y^2 + d$ with $d \in K \subset \mathcal{D}(y^2)$ and thus $\mathcal{D}(y^1) = \mathcal{D}(y^2 + d) \subset \mathcal{D}(y^2)$. □

We obtain a similar result for minimal elements w.r.t. a variable ordering structure. However, we need no condition like (14) or any other transitivity-related assumption.

Theorem 3.2. *Let $K \subset \bigcap_{y \in A} \mathcal{D}(y)$ be a pointed convex cone and denote by $\mathcal{E}_K$ the set of efficient elements of A w.r.t. K and let $\mathcal{E}_K$ be externally stable. Then $\bar{y} \in A$ is a minimal element of A w.r.t. $\mathcal{D}$ if and only if $\bar{y}$ is a minimal element of $\mathcal{E}_K$ w.r.t. $\mathcal{D}$.*

Proof. By Lemma 3.1(c), because of $K \subset \mathcal{D}(y)$ for all $y \in A$ and $\mathcal{E}_K \subset A$, the condition is necessary. To show that it is also sufficient, assume $\bar{y}$ is a minimal element of $\mathcal{E}_K$ but not of A w.r.t. $\mathcal{D}$. Then there exists some $y \in A \setminus \mathcal{E}_K$ with $\bar{y} \in \{y\} + \mathcal{D}(\bar{y}) \setminus \{0\}$. As $\mathcal{E}_K$ is externally stable, there exists some $\hat{y} \in \mathcal{E}_K$ with $y \in \{\hat{y}\} + K \setminus \{0\}$. Then $\bar{y} \in \{\hat{y}\} + K + \mathcal{D}(\bar{y}) \subset \{\hat{y}\} + \mathcal{D}(\bar{y})$, in contradiction to $\bar{y}$ a minimal element of $\mathcal{E}_K$ w.r.t. $\mathcal{D}$. □

Theorem 3.1 and Theorem 3.2 generalize Lemma 2 and Lemma 3 of Hirsch et al. [8] which was given for the special case $K = \mathbb{R}^m_+$ and compact sets A.

We conclude this section with the mentioned scalarization results. First, we consider the nonlinear scalarization functional $\chi_{a,r} \colon \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$,

$$\chi_{a,r}(y) := \inf\{t \in \mathbb{R} \mid a + t\, r - y \in \mathcal{D}(y)\} \quad \text{for all} \quad y \in \mathbb{R}^m.$$

which coincides in the case $\mathcal{D}(y) = K$ for all $y \in \mathbb{R}^m$ with the smallest monotone function, see (11). From the properties of the smallest monotone function we derive that $\chi_{a,r}(y) = \min\{t \in \mathbb{R} \mid a + t\, r - y \in \mathcal{D}(y)\}$ for all $y \in \mathbb{R}^m$ for $r \in \operatorname{int}\left(\bigcap_{y \in A} \mathcal{D}(y)\right)$.

To examine $\chi_{a,r}$ on convexity we need to check whether for all $y^1, y^2 \in \mathbb{R}^m$ and all $\lambda \in (0,1)$ it holds $\chi_{a,r}(y^\lambda) \le \lambda\, \chi_{a,r}(y^1) + (1-\lambda)\, \chi_{a,r}(y^2)$ with $y^\lambda := \lambda\, y^1 + (1-\lambda)\, y^2$. We assume $r \in \operatorname{int}\left(\bigcap_{y \in A} \mathcal{D}(y)\right)$ and set $t^i := \chi_{a,r}(y^i)$, $i = 1, 2$. Then

$$a + t^1\, r - y^1 \in \mathcal{D}(y^1) \qquad \text{and} \qquad a + t^2\, r - y^2 \in \mathcal{D}(y^2). \tag{15}$$

We need to check that $\chi_{a,r}(y^\lambda) \le \lambda\, t^1 + (1-\lambda)\, t^2$, i.e. that $a + (\lambda\, t^1 + (1-\lambda) t^2)\, r - y^\lambda \in \mathcal{D}(y^\lambda)$. This is equivalent to

$$\lambda\, (a + t^1\, r - y^1) + (1-\lambda)\, (a + t^2\, r - y^2) \in \mathcal{D}(y^\lambda). \tag{16}$$

By (15) we have

$$\lambda\, (a + t^1\, r - y^1) + (1-\lambda)\, (a + t^2\, r - y^2) \in \lambda\, \mathcal{D}(y^1) + (1-\lambda)\, D(y^2). \tag{17}$$

Of course, $\lambda\, \mathcal{D}(y^1) = \mathcal{D}(y^1)$ and $(1-\lambda)\, D(y^2) = \mathcal{D}(y^2)$. Comparing (17) with (16) yields that convexity of $\chi_{a,r}$ is given if

$$\lambda\, \mathcal{D}(y^1) + (1-\lambda)\, D(y^2) \subset \mathcal{D}(\lambda\, y^1 + (1-\lambda)\, y^2) \quad \text{for all} \quad \lambda \in (0,1),\ y^1, y^2 \in \mathbb{R}^m,$$

i.e. if $\mathcal{D}$ is convex. However, according to [11] this assumption is equivalent to $\mathcal{D}$ being constant. Thus, the functional $\chi_{a,r}$ can in general not be assumed to be convex.

Theorem 3.3. ***(a)*** *Let $r \in \left(\bigcap_{y \in A} \mathcal{D}(y)\right) \setminus \{0\}$. $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$ if and only if*

$$\chi_{\bar{y},r}(y) > \chi_{\bar{y},r}(\bar{y}) = 0 \quad \textit{for all} \quad y \in A \setminus \{\bar{y}\}.$$

(b) *Let $r \in \operatorname{int}\left(\bigcap_{y \in A} \mathcal{D}(y)\right)$. $\bar{y}$ is a weakly nondominated element of A w.r.t. $\mathcal{D}$ if and only if*

$$\chi_{\bar{y},r}(y) \ge \chi_{\bar{y},r}(\bar{y}) = 0 \quad \textit{for all} \quad y \in A.$$

Proof. (a) First assume $\bar{y}$ to be a nondominated element of A w.r.t. $\mathcal{D}$. As $\mathcal{D}(\bar{y})$ is a pointed convex cone and $r \in \mathcal{D}(\bar{y}) \setminus \{0\}$ it holds $\chi_{\bar{y},r}(\bar{y}) = 0$. If $\bar{y}$ is not a unique minimal solution of $\min_{y \in A} \chi_{\bar{y},r}(y)$ then there exists some $t \in \mathbb{R}$, $t \leq 0$, and some $y \in A \setminus \{\bar{y}\}$ such that $\bar{y} + t\,r - y \in \mathcal{D}(y)$. Because of $t\,r \in -\mathcal{D}(y)$ this implies $\bar{y} \in \{y\} + (\mathcal{D}(y) \setminus \{0\})$ in contradiction to $\bar{y}$ a nondominated element of A w.r.t. $\mathcal{D}$. Next assume $\bar{y}$ is a unique minimizer of $\chi_{\bar{y},r}$ over A. If there is some $y \in A \setminus \{\bar{y}\}$ with $\bar{y} \in \{y\} + \mathcal{D}(y)$ then $\bar{y} + 0 \cdot r - y \in \mathcal{D}(y)$, i.e. $\chi_{\bar{y},r}(y) \leq 0$, which is a contradiction.

(b) First assume $\bar{y}$ to be a weakly nondominated element of A w.r.t. $\mathcal{D}$. If $\bar{y}$ is not a minimal solution of $\min_{y \in A} \chi_{\bar{y},r}(y)$ then there exists some $t \in \mathbb{R}$, $t < 0$, and some $y \in A \setminus \{\bar{y}\}$ such that $\bar{y} + t\,r - y \in \mathcal{D}(y)$. Because of $t\,r \in -\mathrm{int}(\mathcal{D}(y))$ this implies $\bar{y} \in \{y\} + \mathrm{int}(\mathcal{D}(y))$ in contradiction to $\bar{y}$ a weakly nondominated element of A w.r.t. $\mathcal{D}$. Next assume $\bar{y}$ to be a minimizer of $\chi_{\bar{y},r}$ over A. If there is some $y \in A$ with $\bar{y} \in \{y\} + \mathrm{int}(\mathcal{D}(y))$ then there exists some $t < 0$ such that $(\bar{y} - y) + t\,r \in \mathcal{D}(y)$ and hence $\chi_{\bar{y},r}(y) < 0$, which is a contradiction. □

For evaluating whether $\bar{x} \in S$ is an at least weakly nondominated solution of (1) we thus have to check if 0 is the minimal value of the scalar-valued optimization problem

$$\begin{array}{rl} & \min\ t \\ \text{s.t.} & f(\bar{x}) + t\,r - f(x) \in \mathcal{D}(f(x)) \\ & t \in \mathbb{R},\ x \in S \end{array}$$

for some r with $r \in \mathrm{int}(\mathcal{D}(f(x)))$ for all $x \in S$.

For characterizing minimal elements w.r.t. an ordering map $\mathcal{D}$ we can directly apply the function defined in (11). Using Lemma 3.1(b) and (c) and known scalarization results in partially ordered spaces, see for instance [14, 23, 21], we obtain the following necessary and sufficient conditions.

Theorem 3.4. ***(a)*** *If $\bar{y}$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$, then for any pointed convex cone $K \subset \mathcal{D}(\bar{y})$ and any $r \in K \setminus \{0\}$ it holds that $\psi_{\bar{y},r}(y) > \psi_{\bar{y},r}(\bar{y}) = 0$ for all $y \in A \setminus \{\bar{y}\}$.*

(b) *If $\bar{y}$ is a weakly minimal element of A w.r.t. the ordering map $\mathcal{D}$, then for any pointed convex cone $K \subset \mathcal{D}(\bar{y})$ with some $r \in int(K)$ it holds that $\psi_{\bar{y},r}(y) \geq \psi_{\bar{y},r}(\bar{y}) = 0$ for all $y \in A \setminus \{\bar{y}\}$.*

(c) *If for any $a \in Y$, $r \in \mathbb{R}^m$ and any convex cone K with $\mathcal{D}(\bar{y}) \subset K$ it holds that $\psi_{a,r}(y) \geq \psi_{a,r}(\bar{y})$ for all $y \in A$, then $\bar{y}$ is a weakly minimal element of A w.r.t. the ordering map $\mathcal{D}$. If additionally $\psi_{a,r}(y) > \psi_{a,r}(\bar{y})$ for all $y \in A \setminus \{\bar{y}\}$, then $\bar{y}$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$.*

Summarizing the previous results we get for weakly minimal and minimal elements of some set A w.r.t. $\mathcal{D}$ the following complete characterization:

Corollary 3.1. ***(a)*** *Set $K := \mathcal{D}(\bar{y})$ and choose $r \in \mathcal{D}(\bar{y}) \setminus \{0\}$. $\bar{y}$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$ if and only if*

$$\psi_{\bar{y},r}(y) > \psi_{\bar{y},r}(\bar{y}) = 0 \quad \textit{for all} \quad y \in A \setminus \{\bar{y}\}.$$

(b) *Set* $K := \mathcal{D}(\bar{y})$ *and choose* $r \in int(\mathcal{D}(\bar{y}))$. $\bar{y}$ *is a weakly minimal element of* A *w.r.t.* $\mathcal{D}$ *if and only if*

$$\psi_{\bar{y},r}(y) \geq \psi_{\bar{y},r}(\bar{y}) = 0 \;\; \textit{for all} \;\; y \in A.$$

Chen and colleagues studied in [20, 24, 17] for some $r \in \operatorname{int}\left(\bigcap_{y\in\mathbb{R}^m} \mathcal{D}(y)\right)$ the functional

$$(y, z) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto \inf\{t \in \mathbb{R} \mid t\,r - y \in \mathcal{D}(z)\}. \tag{18}$$

In (18) the cone-valued map $\mathcal{D}$ is evaluated in some element z independently of the choice of y while for determining $\chi_{a,r}(y)$ the cones $\mathcal{D}(y)$ are used. In [20, Theorem 3.1] the result of Corollary 3.1(b) is stated but assuming among others $\mathcal{D}$ to be a linear map. Theorem 2.2 in [24] equals Corollary 3.1(b) . Theorem 3.1(ii) in [25] is also very similar to Corollary 3.1(b), but defining $\mathcal{D}$ not on the image space $\mathbb{R}^m$ for the multiobjective optimization problem (1) but by $\mathcal{D}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. In [17], Chen, Yang and Yu relax the assumption $r \in \operatorname{int}\left(\bigcap_{y\in\mathbb{R}^m} \mathcal{D}(y)\right)$ by replacing r in (18) by $r(y)$ with $r := r(y) \in \operatorname{int}(\mathcal{D}(y))$ for all $y \in \mathbb{R}^m$. They study this functional in the context of quasi-vector equilibrium problems.

4 A Numerical Procedure for Continuous Problems

We start with a procedure for determining the nondominated and the minimal elements of an infinite set A which may be, for instance, the image set of the multiobjective optimization problem (1) with a continuous objective function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and $S \subset \mathbb{R}^n$ a nonempty subset given by equality and inequality constraints. The algorithm in this section is mainly based on Lemma 3.1 which states that the set of efficient elements w.r.t. some convex cone K with $K \subset \mathcal{D}(y)$ for all $y \in A$ is a superset of the set of minimal and nondominated elements. Such a cone K always exists, but it may be the trivial cone $K = \{0\}$ and then all elements of A are efficient elements w.r.t. K. In addition to that, many algorithm for determining approximations of the set of efficient elements of A w.r.t. K – we denote this set in the following again by $\mathcal{E}_K$ – require that there exists some $r \in \operatorname{int}(K)$ and for that we assume

$$\operatorname{int}\left(\bigcap_{y\in A} \mathcal{D}(y)\right) \neq \emptyset.$$

As in general not the complete set of efficient elements $\mathcal{E}_K$ w.r.t. K can be determined by such algorithm, we generate an approximation of it. For such an approximation of it many methods can be found in the literature. We use here a procedure introduced in [26], see also [21, 27], which generates an even approximation of the efficient set of the image set of the problem (1) where f needs to satisfy differentiability assumptions and K can be an arbitrary closed convex pointed cone with a nonempty interior. The method uses the scalarization functional given in (11) and determines the parameter a adaptively based on the evaluation of sensitivity information. The method is especially appropriate for determining approximations of the whole efficient set in lower dimensions as $\mathbb{R}^2$. Thereby it can only be guaranteed that weakly efficient approximation points are determined.

We denote the finite set of approximation points of $\mathcal{E}_K$, determined by an arbitrary approximation method as the one mentioned above which delivers at least weakly efficient elements, by $\mathcal{E}_K^{\text{approx}}$. For selecting from $\mathcal{E}_K^{\text{approx}}$ an approximation of the set of optimal (i.e., nondominated or minimal) elements of A w.r.t. $\mathcal{D}$, first the optimal elements of $\mathcal{E}_K^{\text{approx}}$ w.r.t. $\mathcal{D}$ can be selected. For it, a pairwise comparison can be used. In case of a large number of approximation points it may be advantageous to use an algorithm for determining optimal elements of a discrete set as discussed in the next section. By that we reduce the set $\mathcal{E}_K^{\text{approx}}$ to some finite subset $W \subset \mathcal{E}_K^{\text{approx}} \subset A$, respectively. Each element of this subset W is thus an at least weakly efficient element of A and a nondominated or minimal element of $\mathcal{E}_K^{\text{approx}}$ w.r.t. $\mathcal{D}$, respectively. In a second step, we select those elements of W which are nondominated or minimal elements of the set A w.r.t. $\mathcal{D}$, respectively.

4.1 Nondominated Solutions

So far, each element of W is an at least weakly efficient element of A w.r.t. K and a nondominated element of $\mathcal{E}_K^{\text{approx}}$ w.r.t. $\mathcal{D}$. If $\mathcal{E}_K$ is externally stable and assumption (14) of Theorem 3.1 is satisfied, for instance, if $\mathcal{D}$ defines a transitive binary relation, then the nondominated elements of A w.r.t. $\mathcal{D}$ are exactly those efficient elements of A w.r.t. K which are nondominated elements of $\mathcal{E}_K$ w.r.t. $\mathcal{D}$. Based on this theorem the set W may be taken as an approximation of the set of nondominated elements of A w.r.t. $\mathcal{D}$. This approach was followed in [8].

Remark 4.1. *Note that we need that the elements of the set W are nondominated elements of $\mathcal{E}_K$ – and not of $\mathcal{E}_K^{\text{approx}}$ only – w.r.t. $\mathcal{D}$. There might be some element $w \in W$ with $w \notin \{y\} + \mathcal{D}(y) \setminus \{0\}$ for all $y \in \mathcal{E}_K^{\text{approx}}$ but*

$$w \in \{y\} + \mathcal{D}(y) \setminus \{0\} \quad \textit{for some } y \in \mathcal{E}_K \setminus \mathcal{E}_K^{\text{approx}}. \tag{19}$$

For that reason, we use the scalarization results given in Theorem 3.3. Let $r \in \operatorname{int} K$. Then $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$ if and only if

$$\inf\{t \in \mathbb{R} \mid \bar{y} + t\,r - y \in \mathcal{D}(y)\} > 0 \quad \text{for all } \; y \in A \setminus \{\bar{y}\} \tag{20}$$

and $\bar{y}$ is a weakly nondominated element of A w.r.t. $\mathcal{D}$ if and only if

$$\inf\{t \in \mathbb{R} \mid \bar{y} + t\,r - y \in \mathcal{D}(y)\} \geq 0 \quad \text{for all } \; y \in A \setminus \{\bar{y}\}. \tag{21}$$

If we apply a numerical solution method to solve (20) we can in general not verify the strict inequality and thus we can only show the weak nondominatedness of some point $\bar{y}$. If the ordering map has images being representable as Bishop-Phelps cones in the normed space $\mathbb{R}^m$, i.e. if $\mathcal{D}(y) = \{u \in Y \mid \|u\| \leq \ell(y)^\top u\}$ for all $y \in \mathbb{R}^m$ for some map $\ell \colon \mathbb{R}^m \to \mathbb{R}^m$, see (13), then the optimization problem in (21) reads as

$$\text{Minimize } t \text{ subject to } \quad \|\bar{y} + t\,r - y\| - \ell(y)^\top(\bar{y} + t\,r - y) \leq 0,\; t \in \mathbb{R} \;\text{ and }\; y \in A.$$

Due to the norm in the constraint, the above scalar-valued problem is a nondifferentiable nonlinear optimization problem which may even be discontinuous dependently on ℓ. For numerically solving such problems adequate solution methods are necessary.

We sum up the procedure in Algorithm 1.

Algorithm 1 Algorithm for the approximation of the set of nondominated elements

---

Require: set $A \subset Y$, $\mathcal{D}(y)$ for all $y \in A$, $K \subset \bigcap_{y \in A} \mathcal{D}(y)$ a convex cone and $r \in \operatorname{int}(K)$

determine an approximation $\mathcal{E}_K^{\text{approx}} \subset A$ of the set of efficient elements of A with $\mathbb{R}^m$ partially ordered by K

determine the set of all nondominated elements $W =: \{y^1, \ldots, y^k\}$ of $\mathcal{E}_K^{\text{approx}}$ w.r.t. $\mathcal{D}$

put $j = 1$ and $N = \emptyset$

while $j \leq k$ **do**

    determine $\bar{t} = \min\{t \in \mathbb{R} \mid y^j + t\,r - y \in \mathcal{D}(y),\ y \in A\}$

    if $\bar{t} \geq 0$ **then**

        replace N by $N \cup \{y^j\}$

    end if

    replace j by $j + 1$

end while

By Theorem 3.3 we have:

Theorem 4.1. *Let N denote the set generated by Algorithm 1. Then $N \subset A$ and any element of N is a weakly nondominated element of A w.r.t. $\mathcal{D}$.*

4.2 Minimal Solutions

Again, each element of W is an at least weakly efficient element of A w.r.t. K and a minimal element of $\mathcal{E}_K^{\text{approx}}$ w.r.t. $\mathcal{D}$. If $\mathcal{E}_K$ is externally stable, then the minimal elements of A w.r.t. $\mathcal{D}$ are exactly those efficient elements of A w.r.t. K which are minimal elements of $\mathcal{E}_K$ w.r.t. $\mathcal{D}$, see Theorem 3.2. Based on this theorem the set W may be taken as an approximation of the set of minimal elements of A w.r.t. $\mathcal{D}$. Again, this approach was followed in [8], but Remark 4.1 analogously holds for minimal elements. Using the scalarization results of Theorem 3.4, $\bar{y}$ is a weakly minimal element of A w.r.t. the ordering map $\mathcal{D}$ if and only if

$$\inf\{t \in \mathbb{R} \mid \bar{y} + t\,r - y \in \mathcal{D}(\bar{y})\} \geq 0 \quad \text{for all} \quad y \in A \setminus \{\bar{y}\}. \tag{22}$$

This leads to the procedure given in Algorithm 2.

By Corollary 3.1 we have:

Theorem 4.2. *Let M denote the set generated by Algorithm 2. Then $M \subset A$ and any element of M is a weakly minimal element of A w.r.t. $\mathcal{D}$.*

Algorithm 2 Algorithm for the approximation of the set of minimal elements

Require: set $A \subset Y$, $\mathcal{D}(y)$ for all $y \in A$, $K \subset \bigcap_{y \in A} \mathcal{D}(y)$ a convex cone and $r \in \operatorname{int}(K)$

```
determine an approximation $\mathcal{E}_K^{\text{approx}} \subset A$ of the set of efficient elements of $A$
    with $Y$ partially ordered by $K$
determine the set of all minimal elements $W =: \{y^1, \ldots, y^k\}$ of $\mathcal{E}_K^{\text{approx}}$ w.r.t. $\mathcal{D}$
put $j = 1$ and $M = \emptyset$
while $j \leq k$ do
    determine $\bar{t} = \min\{t \in \mathbb{R} \mid y^j + t\,r - y \in \mathcal{D}(y^j),\ y \in A\}$
    if $\bar{t} \geq 0$ then
        replace $M$ by $M \cup \{y^j\}$
    end if
    replace $j$ by $j + 1$
end while
```

5 Numerical Procedure for Discrete Problems

In case of a finite image set $A = f(S)$ of a multiobjective optimization problem the most simple approach for determining all optimal solutions is a pairwise comparison of all elements in A. This may be very time consuming especially if the evaluation of the binary relation $\leq$ is costly. For instance, in [28] an application dealing with a finite set A in the space of Hermitian matrices was considered. The space was assumed to be partially ordered by the cone of positive semidefinite matrices and hence each evaluation of the binary relation corresponds to the determination of the smallest eigenvalue of the difference of two matrices. For that reason numerical methods as the Jahn-Graef-Younes method have been developed for reducing the numerical effort by reducing the number of necessary pairwise comparisons. For $\mathbb{R}^m$ partially ordered by the natural ordering, i.e. $K = \mathbb{R}^m_+$, this procedure was given by Jahn in [29], see also [30, Section 12.4], based on a procedure firstly presented by Younes in [31] and an algorithmic conception by Graef [30, p. 349]. In the following we examine the applicability of this algorithm for variable ordering structures.

It will turn out that in case of a transitive and antisymmetric variable ordering structure the algorithm can directly be applied by replacing the binary relation $\leq_K$ with $K = \mathbb{R}^m_+$ used in its original formulation [30, Section 12.4] by $\leq_1$ or $\leq_2$. However, in case of no transitivity and without antisymmetry the basic algorithm only allows the determination of a superset of the set of nondominated/minimal elements of a set and the algorithm has to be extended by an additional step. Nevertheless the number of pairwise comparisons may be reduced significantly.

We present the algorithm in $\mathbb{R}^m$ but it also directly applies for arbitrary real linear spaces Y with an ordering map $\mathcal{D}\colon Y \rightrightarrows Y$.

5.1 Nondominated Elements

We start by presenting the extended Jahn-Graef-Younes algorithm for the determination of the nondominated elements of a discrete, finite set $A := \{y^1, \ldots, y^k\}$.

The result of the algorithm is discussed in the following theorem. It states that the procedure is well-defined and delivers exactly the set of all nondominated elements of A w.r.t. $\mathcal{D}$. Note that the original Jahn-Graef-Younes method for partially ordered spaces [29] consists only of the first (called forward iteration) and the second (called backward iteration) while-loop, while for variable ordering structures which are not transitive and not antisymmetric the third while-loop (complete comparison for selected elements) has to be added.

Algorithm 3 Jahn-Graef-Younes method for nondominated elements

Require: $A = \{y^1, \ldots, y^k\}$, $\mathcal{D}(y)$ for all $y \in A$

put $U = \{y^1\}$ and $i = 1$
while $i < k$ **do**
  replace i by $i + 1$
  if $y^i \notin \{y\} + \mathcal{D}(y)$ for all $y \in U$ **then**
    replace U by $U \cup \{y^i\}$
  end if
end while
put $\{u^1, \ldots, u^p\} = U$
put $T = \{u^p\}$ and $i = p$
while $i > 1$ **do**
  replace i by $i - 1$
  if $u^i \notin \{u\} + \mathcal{D}(u)$ for all $u \in T$ **then**
    replace T by $\{u^i\} \cup T$
  end if
end while
put $\{t^1, \ldots, t^q\} = T$
put $V = \emptyset$ and $i = 0$
while $i < q$ **do**
  replace i by $i + 1$
  if $t^i \notin \{y\} + \mathcal{D}(y)$ for all $y \in A \setminus T$ **then**
    replace V by $V \cup \{y^i\}$
  end if
end while

Theorem 5.1. *Let A be a finite subset of $\mathbb{R}^m$ and let U, T and V denote the sets gained by Algorithm 3.*

(a) *If $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$, then $\bar{y} \in U$ and $\bar{y} \in T$.*

(b) *The elements of the set $T \subset A$ are all nondominated elements of T w.r.t. $\mathcal{D}$.*

(c) *If $\leq_1$ is a transitive and antisymmetric binary relation, then the set T is exactly the set of all nondominated elements of A w.r.t. $\mathcal{D}$.*

(d) *The set V is exactly the set of all nondominated elements of A w.r.t. $\mathcal{D}$.*

Proof. (a) Assume $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$ but is not in U. Then there exists some $y \in U \subset A$, $y \neq \bar{y}$ with $\bar{y} \in \{y\} + \mathcal{D}(y)$ in contradiction to $\bar{y}$ a nondominated element of the set A w.r.t. $\mathcal{D}$. Next, assume $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$ but is not in T. According to the first part of the proof, $\bar{y} \in U$.

Thus there exists some $y \in T \subset A$, $y \neq \bar{y}$ with $\bar{y} \in \{y\} + \mathcal{D}(y)$ in contradiction to $\bar{y}$ a nondominated element of the set A w.r.t. $\mathcal{D}$.

(b) Let $T =: \{t^1, \ldots, t^q\}$ with $q \leq p \leq k$ and $t^j \in T$ arbitrarily chosen with $1 \leq j \leq q$. We assume the elements of the sets to be ordered in the way they are generated in the algorithm. According to the first while-loop, $t^j \notin \{t^i\} + \mathcal{D}(t^i)$ for all $1 \leq i < j$ and according to the second while-loop, $t^j \notin \{t^i\} + \mathcal{D}(t^i)$ for all $j < i \leq q$. Hence, t^j is a nondominated element of T w.r.t. $\mathcal{D}$.

(c) We first show that for all $y \in A$ there exists a nondominated element $\bar{y}$ of A w.r.t. $\mathcal{D}$ with $y \in \{\bar{y}\} + \mathcal{D}(\bar{y})$. For that, let $y \in A$ be arbitrarily given. If y is a nondominated element of A w.r.t. $\mathcal{D}$ then the assertion is proven. Now, let y be not a nondominated element of A w.r.t. $\mathcal{D}$, i.e. there exists some $y^1 \in A$ with $y^1 \leq_1 y$, $y^1 \neq y$. If y^1 is nondominated we are done. Otherwise there is some $y^2 \neq y^1$ with $y^2 \leq_1 y^1$ and by the transitivity also $y^2 \leq_1 y$, $y^2 \neq y$. If y^2 is not a nondominated element we can find $y^3 \in A \setminus \{y, y^1, y^2\}$ with $y^3 \leq_1 y^2 \leq_1 y^1 \leq_1 y$ and so on. As A is finite and $\leq_1$ is antisymmetric, this procedure stops with a nondominated element $\bar{y} \in A$ of A w.r.t. $\mathcal{D}$ with $\bar{y} \leq_1 y$.

According to (b) all nondominated elements of the set A w.r.t. $\mathcal{D}$ are an element of T and all the elements of T are nondominated elements of T w.r.t. $\mathcal{D}$. It remains to be shown that the elements of T are also nondominated elements of A w.r.t. $\mathcal{D}$. Let $y \in T$ and y be not a nondominated element of A w.r.t. $\mathcal{D}$. Then there exists a nondominated element $\bar{y}$ of A w.r.t. $\mathcal{D}$ with $y \in \{\bar{y}\} + \mathcal{D}(\bar{y}) \setminus \{0\}$. According to (a), $\bar{y} \in T$ in contradiction to y a nondominated element of T w.r.t. $\mathcal{D}$.

(d) This is a direct consequence of (a), (b) and the definition of nondominated elements. $\square$

Conditions ensuring the transitivity and antisymmetry of the binary relation $\leq_1$ are given in Lemma 2.1(c). In that case, the algorithm can be stopped after the second while-loop as the set T consists already of all nondominated elements of A w.r.t. D and thus $T = V$. If the space is partially ordered with some convex cone K, the binary relation $\leq_K$ is transitive and antisymmetric and thus in this case the algorithm can be stopped after the second while loop.

However, without transitivity, the set T does in general not consist of exactly the nondominated elements of the set A w.r.t. $\mathcal{D}$:

Example 5.1. *Let* $A := \{(0,0), (1,0), (0,2)\}$ *and*

$$\begin{aligned} \mathcal{D}(0,0) &= \textit{cone conv}\{(1,-1), (1,1)\}, \\ \mathcal{D}(1,0) &= \textit{cone conv}\{(-1,1), (1,1)\}, \\ \mathcal{D}(0,2) &= \mathbb{R}^2_+ \end{aligned}$$

with cone conv(Ω) *for some set* Ω *the convex cone generated by* Ω. *The unique nondominated element of* A *w.r.t.* $\mathcal{D}$ *is* $(0,0)$ *but Algorithm 3 delivers the sets* $T = U = \{(0,0), (0,2)\}$. *Finally,* $V = \{(0,0)\}$.

Note that one might also use Theorem 3.1, in case the assumptions are satisfied, and the classical Jahn-Graef-Younes method for partially ordered spaces to reduce the numerical effort for determining the nondominated elements w.r.t. a variable ordering structure of a discrete set.

5.2 Minimal Elements

In this section we present an adaption and extension of the classical Jahn-Graef-Younes method for the determination of minimal elements w.r.t. a variable ordering structure. Only the steps 4, 12 and 20 differ to those of Algorithm 3 and thus, only those steps are given explicitly.

Algorithm 4 Jahn-Graef-Younes method for minimal elements (extract)

3: ...
4: **if** $y^i \notin \{y\} + \mathcal{D}(y^i)$ for all $y \in U$ **then**
5:     ...
6: **end if**
11: ...
12: **if** $u^i \notin \{u\} + \mathcal{D}(u^i)$ for all $u \in T$ **then**
13:     ...
14: **end if**
19: ...
20: **if** $t^i \notin \{y\} + \mathcal{D}(t^i)$ for all $y \in A \setminus T$ **then**
21:     ...
22: **end if**
23: ...

Theorem 5.2. *Let A be a finite subset of $\mathbb{R}^m$ and let U, T and V denote the sets gained by Algorithm 4.*

(a) *If $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{D}$, then $\bar{y} \in U$ and $\bar{y} \in T$.*

(b) *The elements of the set $T \subset A$ are all minimal elements of T w.r.t. $\mathcal{D}$.*

(c) *If $\leq_2$ is a transitive and antisymmetric binary relation, then the set T is exactly the set of all minimal elements of A w.r.t. $\mathcal{D}$.*

(d) *The set V is exactly the set of all minimal elements of A w.r.t. $\mathcal{D}$.*

Proof. Analogously to the proof of Theorem 5.1. □

Again, as the following example shows, the elements of T are in general not all also minimal elements of A w.r.t. $\mathcal{D}$ and thus a third while-loop had to be added.

Example 5.2. *Let $A := \{(-1,1),(0,0),(1,0)\}$ and*

$$\begin{aligned} \mathcal{D}(-1,1) &= \mathbb{R}^2_+, \\ \mathcal{D}(0,0) &= \textit{cone conv}\{(1,-1),(1,1)\}, \\ \mathcal{D}(1,0) &= \mathbb{R}^2_+. \end{aligned}$$

The unique minimal element of A w.r.t. $\mathcal{D}$ is $(-1,1)$ but Algorithm 4 delivers the sets $T = U = \{(-1,1),(1,0)\}$. Finally, $V = \{(1,1)\}$.

Note that one might also use Theorem 3.2 and the classical Jahn-Graef-Younes method for partially ordered spaces to reduce the numerical effort for determining the minimal elements w.r.t. a variable ordering structure of a discrete set.

6 Numerical Results

In this chapter we apply the proposed procedures on some test examples. More numerical experiments are provided by Ziegler in his diploma thesis [32].

6.1 Continuous Problems

Example 6.1. ***(Algorithm 1)*** *Consider the set*

$$A = \{(x_1, x_2) \in [0, \pi] \times (0, \pi] \quad | \quad x_1^2 + x_2^2 - 1 - 0.1 \cos\left(16\text{arctan}(\tfrac{x_1}{x_2})\right) \geq 0,$$
$$(x_1 - 0.5)^2 + (x_2 - 0.5)^2 \leq 0.5\} \subset \mathbb{R}^2$$

defined by Tanaka [33]. It holds $\inf_{y \in A} y_i > 0$*,* $i = 1, 2$*. For the variable ordering structure we define the ordering map by* $\mathcal{D}(y) = \{u \in \mathbb{R}^2 \mid \|u\|_2 \leq \ell(y)^\top u\}$ *with*

$$\ell(y) := \frac{2}{\min_{i=1,2} y_i} y \quad \textit{for all } y \in A,$$

compare (12) with $p = (0, 0)$ *and* $\gamma = 0.5$*. We apply Algorithm 1 and first, we determine an approximation* $\mathcal{E}_K^{\text{approx}}$ *of the set of efficient elements of* A *w.r.t.* $K := \mathbb{R}_+^2$*. According to Example 3.1,* $K \subset \mathcal{D}(y)$ *for all* $y \in A$*. The approximation is done using the adaptive parameter control as introduced in [21] with an aimed distance of* 0.02 *between the approximation points. By this procedure the generated approximation points are guaranteed to be at least weakly efficient elements. 55 approximation points are determined shown as dots in Figure 1, left.*

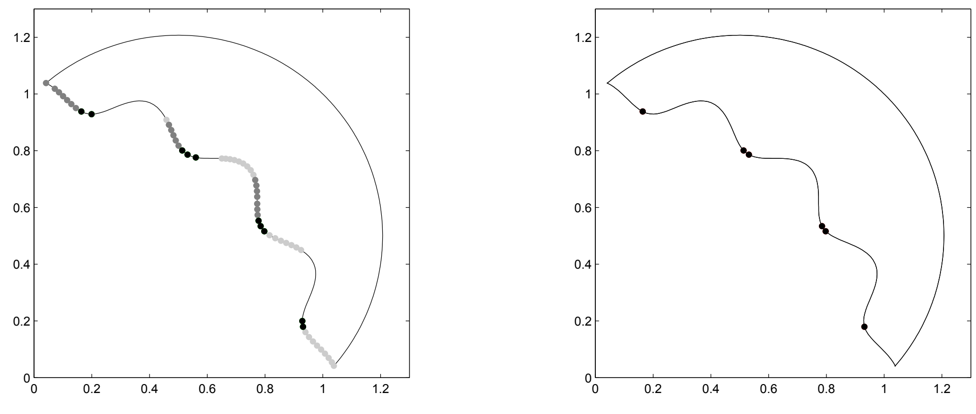

Figure 1: Example 6.1. **Left:** Set $\mathcal{E}_K^{\text{approx}}$. The different colors of the dots are described in the text. **Right:** Approximation of the set of nondominated elements.

Next, Algorithm 3 is applied on the set of approximation points $\mathcal{E}_K^{\text{approx}}$*. After the first while loop, 26 points are deleted (marked in light gray in Figure 1) and only 29 points remain. In the second while loop, 19 points are deleted (marked in dark gray) and 10 points remain. For all of these 10 points (marked in black) it is verified that they are nondominated element of the set* $\mathcal{E}_K^{\text{approx}}$ *w.r.t.* $\mathcal{D}$ *by a complete comparison with all elements in* $\mathcal{E}_K^{\text{approx}}$*.*

For all 10 points y^j, the optimization problem

$$\min\{t \in \mathbb{R} \mid y^j + t\,r - y \in \mathcal{D}(y),\ y \in A\} \tag{23}$$

is solved to check whether they are weakly nondominated elements of A w.r.t. $\mathcal{D}$. This resulted in 6 remaining points (those with zero as minimal value in (23)), see Figure 1, right.

However, note that the optimization problem (23) is not everywhere differentiable due to the definition of $\mathcal{D}$ and it is nonconvex and nonlinear, but in this example we nevertheless just applied a standard numerical solver pre-implemented in Matlab for differentiable optimization problems (using different starting points) which does not guarantee to find global minimal solutions for such optimization problems.

Example 6.2. ***(Algorithm 2)*** *We consider again the set $A \subset \mathbb{R}^2$ of Example 6.1 and the ordering map $\mathcal{D}$ but with*

$$\ell(y) := \frac{2}{\min_{i=1,2}\{y_i + 1.2\}} \left(y + \begin{pmatrix} 1.2 \\ 1.2 \end{pmatrix} \right) \quad \textit{for all } y \in A,$$

compare (12) with $p = (-1.2, -1.2)$ and $\gamma = 0.5$. (For $p = (0, 0)$ as in the previous example no minimal elements are found.)

Applying Algorithm 2, we first determine again an approximation $\mathcal{E}_K^{\text{approx}}$ of the set of efficient elements of A w.r.t. $K := \mathbb{R}^2_+$. This approximation is determined as described in Example 6.1 and consists of 55 approximation points, see Figure 2, left.

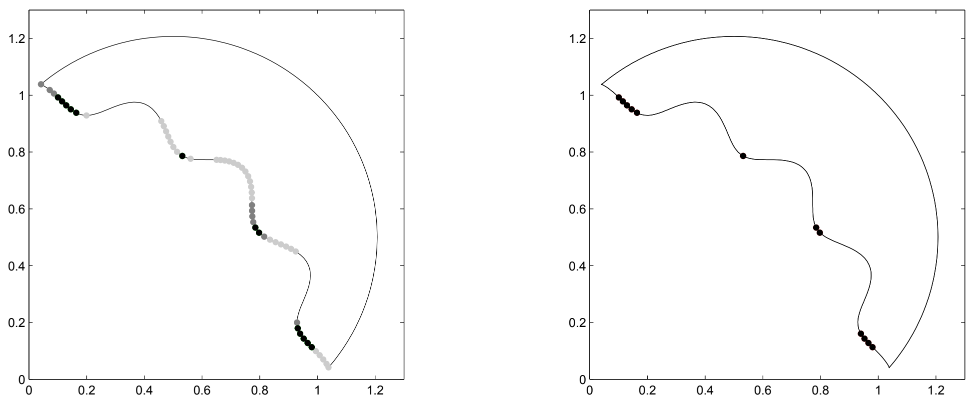

Figure 2: Example 6.2. **Left:** Set $\mathcal{E}_K^{\text{approx}}$. The different colors of the dots are described in the text. **Right:** Approximation of the set of minimal elements.

Next, Algorithm 4 is applied on the set of approximation points $\mathcal{E}_K^{\text{approx}}$. After the first while loop, 33 points are deleted (marked in light gray in Figure 2) and only 22 points remain. In the second while loop, 9 points are deleted (marked in dark gray) and 13 points remain. For all of these 13 points (marked in black) it is verified that they are minimal elements of the set $\mathcal{E}_K^{\text{approx}}$ w.r.t. $\mathcal{D}$ by a complete comparison with all elements in $\mathcal{E}_K^{\text{approx}}$.

For all 13 points y^j, the optimization problem

$$\min\{t \in \mathbb{R} \mid y^j + t\,r - y \in \mathcal{D}(y^j),\ y \in A\}$$

is solved to check whether they are weakly minimal elements of A w.r.t. $\mathcal{D}$. This resulted in 12 remaining points, see Figure 2, right. However, note that the used numerical solver for solving the above scalar-valued problems does not guarantee to find globally optimal solutions.

6.2 Discrete Problems

Example 6.3. ***(Algorithm 3)*** *We reconsider the set $A \subset \mathbb{R}^2$ and the ordering map $\mathcal{D}$ of Example 6.1 and generate a discrete approximation D of this set with 5 014 points by*

$$D := A \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in \{0, 0.01, 0.02, \ldots, \pi\},\ x_2 \in \{0.01, 0.02, \ldots, \pi\}\},$$

compare the set of dots in Figure 3, Left.

The first while-loop of Algorithm 3 selects 27 points (the set U) of the set D as candidates for being nondominated. For that, 61 128 evaluations of the binary relation defined by $\mathcal{D}$ have been necessary. The second while-loop reduces these 27 points to 12 points, the set T, compare Figure 3, right, by only 222 additional evaluations of the binary relation. By comparing these remaining points with all other 5 014 points of the discretization in the third while-loop (additionally, 60 156 evaluations of the binary relation) verifies that these 12 points are exactly the nondominated elements of the discretization set D w.r.t. $\mathcal{D}$. A total of 121 506 evaluations of the binary relation $\leq_1$ are thus needed.

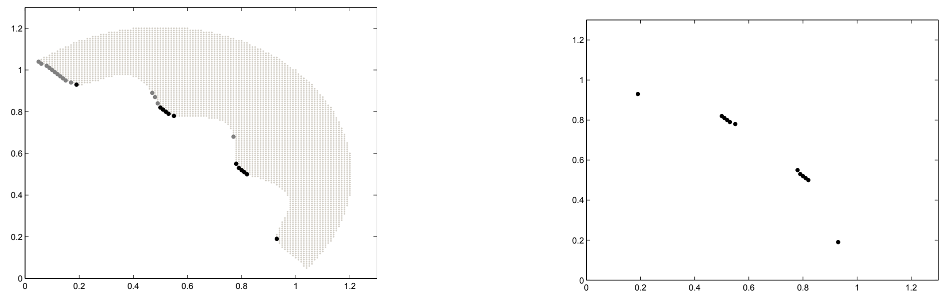

Figure 3: Example 6.3. **Left:** The sets D, U and T in light gray, dark gray and black, respectively. **Right:** Set T.

A pairwise comparison of all 5 014 points with all other points (till it is shown that an element is dominated by another point or nondominated w.r.t. all) needs 4 472 290 evaluations of the binary relation, i.e. a reduction of around 97% is reached.

Example 6.4. ***(Algorithm 4)*** *We reconsider the set $A \subset \mathbb{R}^2$ and the finite set $D \subset A$ as defined in Example 6.3.*

(a) Using the ordering map $\mathcal{D}$ of Example 6.1, the first while-loop of Algorithm 4 selects 18 points. For that only 7 036 evaluations of the binary relation

defined by $\mathcal{D}$ have been necessary. The second while-loop reduces these 18 points to 5 points, the set U, by only 23 additional evaluations of the binary relation. By comparing these remaining points with all other 5 014 points of the discretization in the third while-loop (additionally, 15 060 evaluations of the binary relation) however shows that none of the 5 points of the set U is a minimal element of D w.r.t. $\mathcal{D}$. This is a total of 22 119 evaluations of the binary relation. There are no minimal elements at all in D w.r.t. the variable ordering structure. A pairwise comparison of all 5 014 points with all other elements (till it is shown that an element is not a minimal element w.r.t. $\mathcal{D}$) needs 58 538 evaluations of the binary relation. The algorithm leads thus to a reduction of around 62% of the number of evaluations of $\leq_2$.

(b) *Next, we consider the variable ordering structure defined in Example 6.2. The first while-loop selects now 27 points (the set U) of the set D as candidates for being minimal elements of D w.r.t. $\mathcal{D}$ (8 625 evaluations of the binary relation). The second while-loop reduces to 20 points (213 evaluation), the set T, compare Figure 4, right. By comparing these remaining points with all other 5 014 points of the set D in the third while-loop (100 260 evaluations) verifies that these 20 points are exactly the minimal elements of the discretization set D w.r.t. $\mathcal{D}$.*

This corresponds to a total of 109 098 evaluations of the binary relation. Compared to the number of evaluations needed for a pairwise comparison of all 5 014 points with all other elements till minimality is shown or a preferred element is detected (453 994 evaluations) this is still a reduction of around 76%. In Figure 4, left, all elements of the set D together with the determined minimal elements of D w.r.t. $\mathcal{D}$ and the elements of the sets U and T are shown.

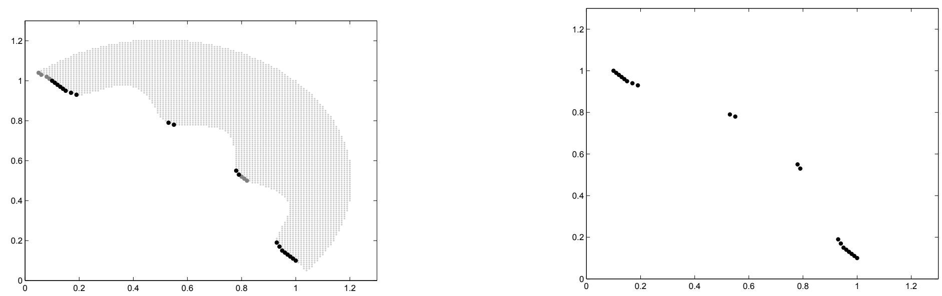

Figure 4: Example 6.4.(b). **Left:** The sets D, U and T in light gray, dark gray and black, respectively. **Right:** Set T.

7 Concluding Remarks

In this manuscript we have presented a numerical procedure for approximating the whole set of nondominated or minimal elements of a continuous multiobjective optimization problem w.r.t. a variable ordering structure. For discrete problems all optimal elements are determined with a reduced effort compared to a pairwise comparison. Note that most of the presented results also apply for vector optimization problems $\min_{x \in S} f(x)$ with $f\colon X \to Y$ and X, Y arbitrary real (topological) linear spaces, S a nonempty subset of X and a cone-valued map $\mathcal{D}\colon Y \rightrightarrows Y$. For more details we refer to [34].

Compared to the procedures proposed in the literature so far, the algorithm for continuous problems of this manuscript allows an approximation of the complete set of optimal elements – and determines not a single optimal solution only as in [4]. Moreover, it can be applied to a broad class of multiobjective optimization problems assuming only the intersection of all cones $\mathcal{D}(y)$ for $y \in A$ to be nontrivial, cf. [4, 7]. The only other numerical approach with the same properties, [8], lacks of guaranteeing that the determined approximation points are indeed optimal solutions while the algorithm presented here guarantees to find at least weakly optimal solutions. The main drawback of the method for continuous problems, which is also a drawback for the method proposed in [8], is the numerical effort caused by first determining approximation points of the efficient set $\mathcal{E}_K$ while later deleting those approximation points of $\mathcal{E}_K$, which are not also optimal w.r.t. the variable ordering structure.

Acknowledgements The author is grateful to Martin Ziegler who performed many numerical experiments with the procedures given in this manuscript in his diploma thesis.

References

1. D. Baatar and M.M. Wiecek, “Advancing equitability in multiobjective programming”, *Comput. Math. Appl.* 52 (2006) 225–234.
2. M.M. Wiecek, “Advances in cone-based preference modeling for decision making with multiple criteria”, *Decision Making in Manufacturing and Services* 1 (2007) 153–173.
3. A. Engau, “Variable preference modeling with ideal-symmetric convex cones”, *J. Global Optim.* 42 (2008) 295–311.
4. M. Wacker, *Multikriterielle Optimierung bei der Registrierung medizinischer Daten* (Diploma thesis, Univ. Erlangen-Nürnberg, 2008).
5. M. Wacker and F. Deinzer, “Automatic robust medical image registration using a new democratic vector optimization approach with multiple measures”, in: *Medical Image Computing and Computer-Assisted Intervention MICCAI 2009*, G.-Z. Yang et al. (Eds.) (2009) 590–597.
6. G. Eichfelder, “Variable ordering structures in vector optimization”, Chapter 4 in: *Recent Developments in Vector Optimization*, Ansari, Q.H., Yao, J.-C. (Eds.) (Springer, Heidelberg, 2011).
7. P.K. Shukla, C. Hirsch and H. Schmeck, “In search of equitable solutions using multi-objective evolutionary algorithms”, in: *Parallel Problem Solving from Nature PPSN XI*, Lecture Notes in Computer Science 6238, R. Schaefer et al. (Eds.) (Springer, Heidelberg, 2011) 687–696.

8. C. Hirsch, P.K. Shukla and H. Schmeck, "Variable preference modeling using multi-objective evolutionary algorithms", in: *Evolutionary Multi-Criterion Optimization - 6th International Conference*, Lecture Notes in Computer Science 6576, R.H.C. Takahashi et al. (Eds.) (Springer, Heidelberg, 2011).
9. Yu, P.L.: Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives. J. Optim. Theory Appl. 14, 319–377 (1974).
10. Chen, G.Y.: Existence of solutions for a vector variational inequality: an extension of the Hartmann-Stampacchia Theorem. J. Optim. Theory Appl. 74, 445–456 (1992).
11. Eichfelder, G.: Cone-valued maps in optimization. Applicable Analysis (2011) DOI: 10.1080/00036811.2011.616499.
12. C. Gerstewitz (Tammer), "Nichtkonvexe Dualität in der Vektoroptimierung", *Wissensch. Zeitschr. TH Leuna-Merseburg* 25 (1983) 357–364.
13. D.T. Luc, *Theory of Vector Optimization* (Springer, Berlin, 1989).
14. A. Pascoletti and P. Serafini, "Scalarizing vector optimization problems", *J. Optim. Theory Appl.* 42 (1984) 499–524.
15. A.M. Rubinov, "Sublinear operators and their applications", *Uspekhi Mat. Nauk* 32 (1977) 113–174.
16. A. Göpfert, H. Riahi, C. Tammer and C. Zălinescu, *Variational Methods in Partially Ordered Spaces* (Springer, New York, 2003).
17. G.Y. Chen, X.Q. Yang and H. Yu, "A nonlinear scalarization function and generalized quasi-vector equilibrium problems", *J. Global Optim.* 32 (2005) 451–466.
18. C. Tammer and C. Zălinescu, "Vector variational principles for set-valued functions", Chapter 11 in: *Recent Developments in Vector Optimization*, Ansari, Q.H., Yao, J.-C. (Eds.) (Springer, Heidelberg, 2011).
19. J. Jahn, "Bishop-Phelps cones in optimization", *International Journal of Optimization: Theory, Methods and Applications* 1 (2009) 123–139.
20. G.Y. Chen and X.Q. Yang, "Characterizations of variable domination structures via nonlinear scalarization", *J. Optim. Theory Appl.* 112 (2002) 97–110.
21. G. Eichfelder, *Adaptive Scalarization Methods in Multiobjective Optimization* (Springer, Heidelberg, 2008).
22. Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization* (Academic Press, London, 1985).
23. P. Weidner, *Ein Trennungskonzept und seine Anwendung auf Vektoroptimierungsverfahren.* (Habilitation thesis, Martin-Luther-Universität Halle-Wittenberg, 1990).
24. G.Y. Chen, X. Huang and X. Yang, *Vector Optimization, Set-Valued and Variational Analysis* (Springer, Berlin, 2005).

25. G. Xiao, H. Xiao and S. Liu, "Scalarization and pointwise well-posedness in vector optimization problems", *J. Global Optim.* 49 (2011) 561–574.

26. G. Eichfelder, *Parametergesteuerte Lösung nichtlinearer multikriterieller Optimierungsprobleme* (Dissertation, University of Erlangen-Nürnberg, 2006).

27. G. Eichfelder, "An adaptive scalarization method in multi-objective optimization", *SIAM J. Optim.* 19 (2009) 1694–1718.

28. G. Eichfelder and M. Gebhardt, "Local specific absorption rate control for parallel transmission by virtual observation points", *Magnetic Resonance in Medicine* (2011) doi 10.1002/mrm.22927.

29. J. Jahn and U. Rathje, "Graef-Younes method with backward iteration", in: *Multicriteria decision making and fuzzy systems - theory, methods and applications*, K.-H. Küfer et al. (Eds.) (Shaker, Aachen, 2006) 75–81.

30. J. Jahn, *Vector Optimization - Theory, Applications, and Extensions*, 2nd edition (Springer, Heidelberg, 2011).

31. Y.M. Younes, *Studies on discrete vector optimization* (Dissertation, University of Demiatta, 1993).

32. M. Ziegler, *Numerische Verfahren zur Bestimmung optimaler Elemente bei variablen Ordnungsstrukturen* (Diploma thesis, Univ. Erlangen-Nürnberg, 2012).

33. M. Tanaka, "GA-based decision support system for multi-criteria optimization", *Proceedings of the international Conference on Systems, Man and Cybernetics 2* (1995) 1556–1561.

34. G. Eichfelder, *Variable Ordering Structures in Vector Optimization* (Habilitation thesis, Univ. Erlangen-Nürnberg, 2011).